MATH 2040 Lecture 19 (14/11/2016)

§ Unitary and Orthogonal operators

Setting: $(V, \langle , \rangle)$ over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ ($\dim V < +\infty$)

Idea: What are the transformations preserving the structures of ① vector space $+, \cdot$ (linear) ② inner product $\langle , \rangle$ (unitary / orthogonal) ($\mathbb{C}$ / $\mathbb{R}$)

Def$^{\underline{n}}$: A linear operator $T: V \to V$ is unitary / orthogonal ($\mathbb{F} = \mathbb{C}$) ($\mathbb{F} = \mathbb{R}$) if $\boxed{\|Tx\| = \|x\| \quad \forall x \in V}$

Thm: TFAE

(a) $\|Tx\| = \|x\| \quad \forall x \in V$

(b) $T^*T \underset{(*)}{=} TT^* = I$ (i.e. (*) means normal, $T^* = T^{-1}$)

(c) $\langle Tx, Ty \rangle = \langle x, y \rangle \quad \forall x, y \in V$

(d) β O.N.B $\Rightarrow$ $T(\beta)$ O.N.B.

(e) $\exists$ some O.N.B. β s.t. $T(\beta)$ O.N.B.

Proof: (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\underset{\text{easy}}{\Rightarrow}$ (d) $\underset{\text{easy}}{\Rightarrow}$ (e) $\Rightarrow$ (a)

$\boxed{(a) \Rightarrow (b)}$: (a) $\Rightarrow$ $\forall x \in V$

$$\langle x, x \rangle = \|x\|^2 \overset{(a)}{=} \|Tx\|^2 = \langle Tx, Tx \rangle = \langle x, T^*Tx \rangle$$

$$\Rightarrow \quad ^{①} \langle x, \underbrace{(I - T^*T)}_{\underline{Q}:\ =0\,?} x \rangle = 0 \quad \forall x \in V$$

<u>Observe</u>: $^{②} I - T^*T$ is self adjoint

<u>Why?</u> $(I - T^*T)^* = I^* - T^*T^{**} = I - T^*T$

<u>Lemma</u>: ① $\langle x, Ux \rangle = 0 \ \forall x \in V$
② U self adjoint
$\Big\} \Rightarrow U = 0$

<u>Ex</u>: ② cannot be dropped, give a counterexample.

<u>Pf of Lemma</u>: ② $\Rightarrow$ U diagonalizable

λ e-value of U w/. e-vector $v \neq 0$ $\Rightarrow$ $0 \overset{①}{=} \langle v, Uv \rangle = \langle v, \lambda v \rangle = \bar{\lambda} \|v\|^2$

$\Rightarrow \lambda = 0$.

So, $U = 0$.

□

So, Lemma $\Rightarrow$ $T^*T = I$ <u>done</u>! ($\because \dim V < +\infty$).

$\boxed{(b) \Rightarrow (c)}$: Assume $T^*T = I = TT^*$.

$$\langle Tx, Ty \rangle = \langle x, T^*Ty \rangle = \langle x, y \rangle \quad \forall x, y \in V.$$

$\boxed{(e) \Rightarrow (a)}$: Let $\beta = \{v_1, \ldots, v_n\}$ O.N.B.

s.t. $T(\beta) = \{Tv_1, \ldots, Tv_n\}$ O.N.B.

Take any $v \in V$. $\exists ! a_i \in \mathbb{F}$, s.t.

$$v = a_1 v_1 + \cdots + a_n v_n \overset{\beta \text{ o.n.B.}}{\Longrightarrow} \|v\|^2 = \sum_{i=1}^{n} |a_i|^2$$

$$\|$$

$$Tv = a_1 Tv_1 + \cdots + a_n Tv_n \overset{T(\beta) \text{ o.n.B}}{\Longrightarrow} \|Tv\|^2 = \sum_{i=1}^{n} |a_i|^2$$

$\blacksquare$

Lemma: $T: V \to V$ unitary / orthogonal

$\Rightarrow$ ① T normal

② All eigenvalues $\lambda \in \mathbb{F}$ have $|\lambda| = 1$.

Proof: ① proved above as (b).

② $Tv = \lambda v \Rightarrow \|v\| = \|Tv\| = \|\lambda v\| = |\lambda| \, \|v\|$

$\blacksquare$

Cor: $(\mathbb{F} = \mathbb{C})$ Ex: what about $\mathbb{F} = \mathbb{R}$?

$$\left.\begin{array}{l} ① \ T \text{ normal} \\ ② \text{ All eigenvalues } \lambda \in \mathbb{F} \text{ have } |\lambda| = 1. \end{array}\right\} \underset{(\Leftarrow)}{\Rightarrow} T \text{ unitary}$$

Proof: T normal $\Rightarrow$ By Spectral Decomposition,

Recall: (a) $V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$ & $E_{\lambda_i} \perp E_{\lambda_j}$ $(i \neq j)$

(b) $T = \lambda_1 T_1 + \cdots + \lambda_k T_k$ & $T_i T_j = T_j T_i = 0$ $(i \neq j)$

$T_i^* = T_i$

To show T is unitary, we need to prove

$$T T^* = I$$

$$T = \lambda_1 T_1 + \cdots + \lambda_k T_k$$

$$T^* = \bar{\lambda}_1 T_1^* + \cdots + \bar{\lambda}_k T_k^* = \bar{\lambda}_1 T_1 + \cdots + \bar{\lambda}_k T_k$$

$$TT^* = (\lambda_1 T_1 + \cdots + \lambda_k T_k)(\bar{\lambda}_1 T_1 + \cdots + \bar{\lambda}_k T_k)$$

$$= |\lambda_1|^2 T_1^2 + \cdots + |\lambda_k|^2 T_k^2 \qquad (T_i T_j = 0 \text{ if } i \neq j)$$

$$= T_1 + \cdots + T_k \overset{(a)}{=} I \qquad (|\lambda_i| = 1,\ T_i^2 = T_i)$$

$\square$

Cor: $(\mathbb{F} = \mathbb{C})$ Assume T normal.

T self adjoint $\iff$ All eigenvalues of $T \in \mathbb{R}$.

Pf: "$\Rightarrow$" shown before.

"$\Leftarrow$" By Spectral Decomposition,

$$T = \lambda_1 T_1 + \cdots + \lambda_k T_k \quad \text{where } \lambda_i \in \mathbb{R}$$

$$\Rightarrow \quad T^* = \bar{\lambda}_1 T_1^* + \cdots + \bar{\lambda}_k T_k^*$$

$$= \lambda_1 T_1 + \cdots + \lambda_k T_k = T$$

self-adjoint!

$\square$

Q: What about the "matrix" side?

Recall: T unitary / orthogonal $\xRightarrow[\text{O.N.B.}]{\beta}$ $A = [T]_\beta$

i.e. $T^*T = I = TT^*$ $\qquad$ $A^*A = I = AA^*$

Defⁿ: $A \in M_{n\times n}(\mathbb{F})$ $\qquad \mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

- A unitary $(\mathbb{F} = \mathbb{C})$ $\iff$ $A^*A = I = AA^*$
- A orthogonal $(\mathbb{F} = \mathbb{R})$ $\iff$ $A^tA = I = AA^t$

Lemma: β, γ O.N.B for $(V, \langle\cdot,\cdot\rangle)$

$\Rightarrow$ $Q = [I]_\beta^\gamma$ unitary / orthogonal
$\qquad\qquad\qquad\qquad \mathbb{F} = \mathbb{C} \qquad \mathbb{F} = \mathbb{R}$

Why? $\mathbb{F} = \mathbb{R}$ $\qquad Q$ orthogonal $\iff Q^tQ = I$

$V = \mathbb{R}^n$ $\qquad \gamma$: std basis ONB

$\qquad\qquad\qquad \beta = \{v_1, \dots, v_n\}$ ONB

$$Q = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \Rightarrow Q^t = \begin{pmatrix} — & v_1 & — \\ & \vdots & \\ — & v_n & — \end{pmatrix}$$

$$Q^tQ = \begin{pmatrix} — & v_1 & — \\ & \vdots & \\ — & v_n & — \end{pmatrix}\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} = \begin{pmatrix} & & \\ & \langle v_i, v_j\rangle & \\ & & \end{pmatrix} \overset{?}{=} I$$

holds $\iff$
β O.N.B

$\blacksquare$

Cor: β, γ O.N.B. $T: V \to V$ linear

$$\Rightarrow \quad [T]_\beta = Q^{-1}[T]_\gamma Q \overset{(*)}{=} Q^*[T]_\gamma Q$$

same

Def$^{\underline{n}}$: $A, B \in M_{n\times n}(\mathbb{F})$ are unitarily / orthogonally equivalent

if $\exists$ unitary / orthogonal matrix $Q \in M_{n\times n}(\mathbb{F})$

s.t. $\boxed{A = Q^* B Q}$

Rephrase theorems in matrix form: $A \in M_{n\times n}(\mathbb{F})$

Spectral Thm:

A unitarily / orthogonally equivalent to a diagonal matrix $\Leftrightarrow$ A normal / self-adjoint

Schur Lemma:

char. poly of A splits / $\mathbb{F}$ $\Rightarrow$ A unitarily / orthogonally equivalent to an upper triangular matrix

E.g. (Ex.)

$$A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix} \in M_{3\times 3}(\mathbb{R}) \quad \text{symmetric}$$

$\Rightarrow \exists$ orthogonal Q s.t. $\underset{\underset{Q^{-1}}{\parallel}}{Q^t} A Q$ = diagonal.

$\square$